The Vlasov-Poisson system of equations under inversion in a sphere

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(Dated: 28 November 2024)

In this work we develop the transformation of the Vlasov-Poisson's system of equations under inversion in a sphere. We depart by transforming the microscopic quantities and subsequently derive the transformed statistical functions. We apply the so transformed expressions to obtain the inverted average plasma potential for a plasma system where the velocity distribution is a Boltzmann distribution. An expression for the induced charge on the sphere follows naturally from the expression of the inverted electrostatic field, which furnishes a mean to distinguish the induced charge number by the plasma on the sphere from its total charge.

I. INTRODUCTION

The method of image charges of electrostatics is well known among physicists since their graduating education, where a typical use case is to obtain the electrostatic potential around a spherical conductor surrounded by few charged particles. It is a mapping, generally used on a system of equations in order to simplify or to allow a mathematical solution to a problem in some reciprocal space, or at least to provide one alternative path to achieve an analytical or numerical solution. The generalisation of the method from few particles to a microscopic distribution of a large number of charges can be achieved through the so called transformation *inversion in a sphere*[1], which is given in vectorial form by[2]

$$\mathbf{r}' = \frac{a^2}{r}\hat{r} = \frac{a^2}{r^2}\mathbf{r} \,, \tag{1}$$

where a is the radius of the sphere, $\mathbf{r}$ is the position vector and $\mathbf{r}'$ is the reciprocal position vector. Such a transformation is well known from classical electrodynamic theory.

In the present work we are concerned to apply this transformation to obtain the Vlasov-Poisson's system of equations, by taking ensemble averages over the transformed microscopic distributions and fields. We hope that the equations obtained in the present work will be useful to solve electrostatic plasma physics problems. An advantage of the technique of projecting the image of a plasma into the other side of an interface[3,4] is that it allows the integration of the Vlasov-Poisson's equations to be performed in the whole space and, in the case of a spherical interface, by replacing the integration of the Vlasov-Poisson's equations in the range $r < a$ by their transformed equations. The applicability of the transformed equations in this work is limited to collisionless plasmas and electrostatic fields. We can follow two ways to approach a problem: the first is to replace the original system by the transformed one and obtain a solution for this one. The second is to use the transformed equations to extend the plasma inside the sphere, so that the integration can be made in the whole space. In Fig. 1 we can see the picture of a finite plane, which can represent a plasma layer, and its image inside an unitary sphere.

The manuscript is divided as follows: we develop the transformation on the microscopic functions in Section II, where the transformation on the velocities is introduced, and the transformed exact dynamical equation, which is known on the field of plasma physics as the Klimontovich equation, is obtained. From the transformed microscopic functions we derive the transformed statistical functions and present the final system of electrostatic equations, in Section III. In Section IV we give an example of a stationary solution for the inverted one particle distribution function, and apply the transformed equations to one example of the plasma-sphere system, where we obtain not only the electrostatic potential around the sphere but also determine the induced charge on the sphere, and also determine the plasma stationary states. In Section V we make our final remarks about the results and the present work.

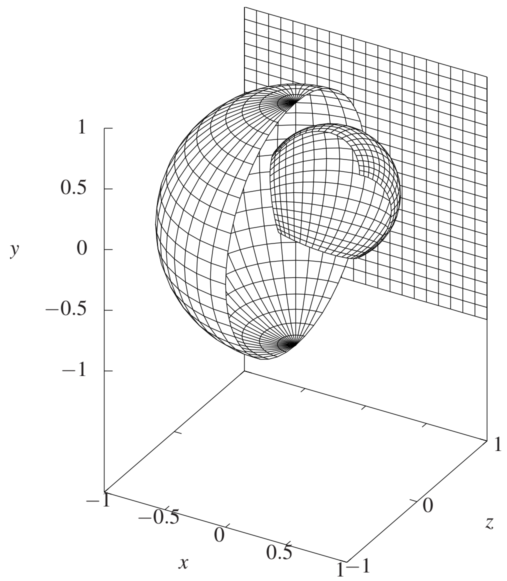

FIG. 1. The plane described by the position vector $\mathbf{r} = x\hat{x} + y\hat{y} + \hat{z}$ in the range $-1 \le x \le 1$, $-1 \le y \le 1$, which is tangent to a sphere of unitary radius, and its inverted image inside the sphere. The sphere is cut in half for a good visualisation of the transformed plane.

II. MICROSCOPIC FUNCTIONS

The Poisson's equation

$$\nabla^2 \phi = -4\pi\rho(\mathbf{r}) \tag{2}$$

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is invariant under the transformation (1). This means that, if $\phi(r)$ is the exact potential due to a microscopic distribution of charges, then the potential should transform like[1]

$$\phi'(r) = \frac{a}{r}\phi(\frac{a^2}{r}) = \frac{r'}{a}\phi(r'), \tag{3}$$

which is the Kelvin transformation[5] of ϕ. Indeed, by direct partial derivation of (3), we obtain

$$\begin{aligned}&\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\phi'}{\partial r}\right]\\ &= \frac{r'^4}{a^5}\left(2\frac{\partial\phi}{\partial r'} + r'\frac{\partial^2\phi}{\partial r'^2}\right) = \frac{r'^5}{a^5}\frac{1}{r'}\frac{\partial^2}{\partial r'^2}\left[r'\phi\right]\\ &= \frac{r'^5}{a^5}\frac{1}{r'^2}\frac{\partial}{\partial r'}\left[r'^2\frac{\partial\phi}{\partial r'}\right] = \frac{a^5}{r^5}\left(\frac{1}{r'^2}\frac{\partial}{\partial r'}\left[r'^2\frac{\partial\phi}{\partial r'}\right]\right).\end{aligned} \tag{4}$$

From Poisson's equation we arrive to $\rho'(r) = (a^5/r^5)\rho(a^2/r)$ where $r' = a^2/r$.

A. Equations of motion

If $\mathbf{r}_i(t)$ is the position vector of a plasma particle, then it follows from (1) that its image has a position $\mathbf{r}'_i(t)$ given by

$$\mathbf{r}'_i = \frac{a^2}{r_i^2}\mathbf{r}_i. \tag{5}$$

It follows from (5) that

$$\begin{aligned}\frac{d\mathbf{r}'_i}{dt} &= \mathbf{v}'_i = \frac{a^2}{r_i^2}\mathbf{v}_i - 2\frac{a^2}{r_i^2}\left(\frac{\mathbf{r}_i\cdot\mathbf{v}_i}{r_i^2}\right)\mathbf{r}_i,\\ \frac{d\mathbf{v}'_i}{dt} &= \mathbf{a}'_i = \frac{a^2}{r_i^2}\mathbf{a}_i - 4\frac{a^2}{r_i^2}\left(\frac{\mathbf{r}_i\cdot\mathbf{v}_i}{r_i^2}\right)\mathbf{v}_i\\ &\quad + 8\frac{a^2}{r_i^2}\left(\frac{\mathbf{r}_i\cdot\mathbf{v}_i}{r_i^2}\right)^2\mathbf{r}_i - 2\frac{a^2}{r_i^2}\left(\frac{v_i^2 + \mathbf{r}_i\cdot\mathbf{a}_i}{r_i^2}\right)\mathbf{r}_i,\end{aligned} \tag{6}$$

where $\mathbf{v}_i = d\mathbf{r}_i/dt$ and $\mathbf{a}_i = d\mathbf{v}_i/dt$.

B. The microscopic distribution in spherical coordinates

We need to transform the microscopic particle distribution[6]

$$N_\alpha(\mathbf{r},\mathbf{v},t) = \sum_i \delta(\mathbf{r} - \mathbf{r}_{\alpha i}(t))\delta(\mathbf{v} - \mathbf{v}_{\alpha i}(t)). \tag{7}$$

In this work the index α identifies the type of plasma particle, while the index i identifies a particle of type α, represented by αi. From here to the remaining of the Section II, we omit the index α in summations for economy of notation.

The position distribution in spherical coordinates is

$$\delta(\mathbf{r} - \mathbf{r}_i) = \frac{1}{r_i^2\sin\theta_i}\delta(r - r_i)\delta(\theta - \theta_i)\delta(\varphi - \varphi_i). \tag{8}$$

We want to obtain the microscopic velocity distribution in spherical generalized coordinates.

Let us obtain the Jacobian of the transformation $dv_x dv_y dv_z \to |J| d\dot{r} d\dot{\theta} d\dot{\varphi}$. For this we use the well known coordinate transformation $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$. By taking the total time derivative of x, y and z,

$$\begin{aligned}\dot{x} &= \dot{r}\sin\theta\cos\varphi + r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi,\\ \dot{y} &= \dot{r}\sin\theta\sin\varphi + r\dot{\theta}\cos\theta\sin\varphi + r\dot{\varphi}\sin\theta\cos\varphi,\\ \dot{z} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta,\end{aligned}$$

from which the Jacobian of the transformation is $J = r^2\sin^3\theta\cos^2\varphi + r^2\cos^2\theta\sin\theta\cos^2\varphi + r^2\sin\theta\sin^2\varphi = r^2\sin\theta$. Therefore the velocity distribution in spherical coordinates is given by

$$\delta(\mathbf{v} - \mathbf{v}_i) = \frac{1}{r_i^2|\sin\theta_i|}\delta(\dot{r} - \dot{r}_i)\delta(\dot{\theta} - \dot{\theta}_i)\delta(\dot{\varphi} - \dot{\varphi}_i). \tag{9}$$

From (9) and (8), the microscopic distribution (7) becomes

$$N_\alpha(r,\theta,\varphi,\dot{r},\dot{\theta},\dot{\varphi},t) = \sum_i \frac{1}{r_i^2\sin\theta_i}\delta(r - r_i)\delta(\theta - \theta_i)\delta(\varphi - \varphi_i)\frac{1}{r_i^2\sin\theta_i}\delta(\dot{r} - \dot{r}_i)\delta(\dot{\theta} - \dot{\theta}_i)\delta(\dot{\varphi} - \dot{\varphi}_i), \tag{10}$$

where we have written $|\sin\theta_i| = \sin\theta_i$ because $0 \le \theta_i \le \pi$. The microscopic distribution (10) is in an appropriate form, so that we can apply the transformation (1).

1. The inverted position distribution

The inversion of the microscopic position distribution is given in Ref.[1]. In this subsection we make a quick review by scrutinizing the development of such transformation considering the charge distribution

$$\rho_\alpha(r,\theta,\varphi) = \sum_i q_i\delta(\Omega - \Omega_i)\frac{1}{r_i^2}\delta(r - r_i), \tag{11}$$

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where $\delta(\Omega-\Omega_i) = (1/\sin\theta_i)\delta(\theta-\theta_i)\delta(\varphi-\varphi_i)$. By making the substitution $r \to \frac{a^2}{r}$, the Delta function transforms to

$$\delta(\frac{a^2}{r} - r_i) = \delta(g(r)).$$

By finding the root r_0 of $g(r)$,

$$\frac{a^2}{r} - r_i = 0 \quad \Rightarrow \quad r_0 = \frac{a^2}{r_i},$$

we can write

$$\begin{aligned}\delta(\frac{a^2}{r} - r_i) &= \frac{\delta(r-r_0)}{\left|-\frac{a^2}{r^2}\right|_{r=r_0}} = \frac{r_0^2}{a^2}\delta(r-r_0) \\ &= \frac{a^2}{r_i^2}\delta(r-\frac{a^2}{r_i}) = \frac{r^2}{a^2}\delta(r-\frac{a^2}{r_i}).\end{aligned}$$

By forming the product

$$\frac{1}{r_i^2}\delta(\frac{a^2}{r} - r_i) = \frac{a^2}{r_i^4}\delta(r-\frac{a^2}{r_i}) = \frac{a^6}{r_i^6}\frac{\delta(r-\frac{a^2}{r_i})}{\frac{a^4}{r_i^2}} \tag{12}$$

the charge distribution (11) becomes

$$\rho_\alpha(\frac{a^2}{r}, \theta, \varphi) = \sum_i q_i \delta(\Omega-\Omega_i)\frac{a^6}{r_i^6}\frac{\delta(r-\frac{a^2}{r_i})}{\frac{a^4}{r_i^2}}$$

so that

$$\begin{aligned}\rho'_\alpha(r,\theta,\varphi) &= \left(\frac{a^5}{r^5}\right)\sum_i \frac{q_i a}{r_i}\frac{a^5}{r_i^5}\delta(\Omega-\Omega_i)\frac{\delta(r-\frac{a^2}{r_i})}{\frac{a^4}{r_i^2}} \\ &= \sum_i q'_i \delta(\mathbf{r}-\mathbf{r}'_i)\end{aligned}$$

where $q'_i = \frac{a}{r_i} q_i$.

2. The inverted velocity distribution

By following the same development of Subsection II B 1, we transform the microscopic velocity distribution (9):

$$\delta(\dot{\overline{\frac{a^2}{r}}} - \dot{r}_i) = \delta(-\frac{a^2}{r^2}\dot{r} - \dot{r}_i) = \delta(\frac{a^2}{r^2}\dot{r} + \dot{r}_i) = \delta(g(\dot{r})).$$

The zero of $g(\dot{r})$ is

$$\frac{a^2}{r^2}\dot{r} + \dot{r}_i = 0 \quad \Rightarrow \quad \dot{r}_0 = -\frac{r^2}{a^2}\dot{r}_i,$$

so that

$$\delta(\dot{\overline{\frac{a^2}{r}}} - \dot{r}_i) = \frac{\delta(\dot{r}-\dot{r}_0)}{\left|\frac{a^2}{r^2}\right|_{\dot{r}=\dot{r}_0}} = \frac{r^2}{a^2}\delta(\dot{r}-\dot{r}_0) = \frac{r^2}{a^2}\delta(\dot{r}+\frac{r^2}{a^2}\dot{r}_i). \tag{13}$$

C. The inverted microscopic distribution

By applying the transformation (1) to (8) and using (12), we obtain the inverted position distribution

$$\frac{a^6}{r_i^6}\frac{1}{\frac{a^4}{r_i^2}\sin\theta_i}\delta(r-\frac{a^2}{r_i})\delta(\theta-\theta_i)\delta(\varphi-\varphi_i).$$

By applying the transformation (1) to (9) and using (13), we obtain the inverted velocity distribution

$$\frac{1}{\frac{a^4}{r_i^2}\sin\theta_i}\frac{r^2}{a^2}\delta(\dot{r}+\frac{r^2}{a^2}\dot{r}_i)\delta(\dot{\theta}-\dot{\theta}_i)\delta(\dot{\varphi}-\dot{\varphi}_i).$$

Therefore, $N_\alpha(\frac{a^2}{r})$ is given by

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$$N_\alpha(\frac{a^2}{r},\theta,\varphi,\dot r,\dot\theta,\dot\varphi,t)=\sum_i \frac{a^6}{r_i^6}\frac{1}{\frac{a^4}{r_i^2}\sin\theta_i}\delta(r-\frac{a^2}{r_i})\delta(\theta-\theta_i)\delta(\varphi-\varphi_i)\frac{a^6}{r_i^6}\frac{1}{\frac{a^4}{r_i^2}\sin\theta_i}\delta(\dot r+\frac{a^2}{r_i^2}\dot r_i)\delta(\dot\theta-\dot\theta_i)\delta(\dot\varphi-\dot\varphi_i)$$
$$=\sum_i\frac{a^{12}}{r_i^{12}}\delta(\mathbf{r}-\mathbf{r}'_i)\delta(\mathbf{v}-\mathbf{v}'_i)=\sum_i\frac{r_i'^{12}}{a^{12}}\delta(\mathbf{r}-\mathbf{r}'_i)\delta(\mathbf{v}-\mathbf{v}'_i)=\frac{r^{12}}{a^{12}}\sum_i\delta(\mathbf{r}-\mathbf{r}'_i)\delta(\mathbf{v}-\mathbf{v}'_i), \tag{14}$$

where

$$\delta(\mathbf{r}-\mathbf{r}'_i)\delta(\mathbf{v}-\mathbf{v}'_i)=\frac{1}{\frac{a^4}{r_i^2}\sin\theta_i}\delta(r-\frac{a^2}{r_i})\delta(\theta-\theta_i)\delta(\varphi-\varphi_i)\frac{1}{\frac{a^4}{r_i^2}\sin\theta_i}\delta(\dot r+\frac{a^2}{r_i^2}\dot r_i)\delta(\dot\theta-\dot\theta_i)\delta(\dot\varphi-\dot\varphi_i).$$

Equation (14) is the microscopic distribution of the image particles located at $\mathbf{r}'_i$ with velocity $\mathbf{v}'_i$, where

$$\mathbf{r}'_i=\left(\frac{a^2}{r_i},\theta_i,\varphi_i\right),\qquad \mathbf{v}'_i=\left(-\frac{a^2}{r_i^2}\dot r_i,\dot\theta_i,\dot\varphi_i\right). \tag{15}$$

The consistency of Eqs. (15) with (5) and (6) can be readily verified.

From (14) we define the inverted microscopic distribution N'_α,

$$N_\alpha(\frac{a^2}{r},\theta,\varphi,\dot r,\dot\theta,\dot\varphi,t)=\frac{r^{12}}{a^{12}}\sum_i\delta(\mathbf{r}-\mathbf{r}'_i)\delta(\mathbf{v}-\mathbf{v}'_i)=\frac{r^{12}}{a^{12}}N'_\alpha(r,\theta,\varphi,\dot r,\dot\theta,\dot\varphi,t). \tag{16}$$

D. The inverted Klimontovich equation

The Klimontovich equation is known in the plasma physics literature[6] as a conservation law in the phase space. By taking the total time derivative of (16) we obtain the inverted Klimontovich equation

$$\frac{a^{12}}{r^{12}}\frac{d}{dt}\left[N_\alpha(\frac{a^2}{r},\theta,\varphi,\dot r,\dot\theta,\dot\varphi,t)\right]=\frac{d}{dt}\left[N'_\alpha(r,\theta,\varphi,\dot r,\dot\theta,\dot\varphi,t)\right]+12\frac{\dot r}{r}N'_\alpha(r,\theta,\varphi,\dot r,\dot\theta,\dot\varphi,t)=0. \tag{17}$$

It follows from (17) and (16) that the total time derivative of $N'_\alpha(r)$ is formally the same as that of $N_\alpha(r)$, but with $\mathbf{r}_i$ and $\mathbf{v}_i$ replaced by $\mathbf{r}'_i$ and $\mathbf{v}'_i$ respectively.

1. The inverted electrostatic radial field

To obtain an expression for the inverted force, we need to know how the field transforms under inversion. Using the derivatives of Section II, the field

$$E_r(r)=-\frac{\partial\phi}{\partial r}$$

transforms according to

$$E_r(r')=-\frac{\partial\phi(r')}{\partial r'}=-\frac{\partial r}{\partial r'}\frac{\partial}{\partial r}\left[\frac{r}{a}\phi'(r)\right]=\frac{r^2}{a^2}\left(\frac{1}{a}\phi'(r)+\frac{r}{a}\frac{\partial\phi'(r)}{\partial r}\right)=E'_r(r), \tag{18}$$

where [1]

$$E'_r(r)=\frac{r^2}{a^2}\frac{\partial}{\partial r}\left[\frac{r}{a}\phi'(r)\right]. \tag{19}$$

[1] The definition of Eq. (19) as the electric field comes from the inverted Lagrangian of a particle. If $L(\mathbf{r}_i,\mathbf{v}_i)=\frac{m_i}{2}v_i^2-q_i\phi(r_i)$ then $L(\mathbf{r}'_i,\mathbf{v}'_i)=\frac{m_i}{2}\frac{a^4}{r_i^4}v_i^2-q_i\frac{r_i}{a}\phi'(r_i)=L'(\mathbf{r}_i,\mathbf{v}_i)$ is the inverted Lagrangian. Therefore,

$$\frac{d}{dt}\frac{\partial L'}{\partial\dot r_i}-\frac{\partial L'}{\partial r_i}=-4m_i\frac{a^4}{r_i^4}\frac{\dot r_i^2}{r_i}+m_i\frac{a^4}{r_i^4}\ddot r_i+2m_i\frac{a^4}{r_i^4}\frac{v_i^2}{r_i}-m_i\frac{a^4}{r_i^4}\frac{r_i^2\dot\theta_i^2+r_i^2\dot\varphi^2\sin^2\theta_i}{r_i}+q_i\frac{\partial}{\partial r_i}\left[\frac{r_i}{a}\phi'(r_i)\right]=0$$

and $\ddot r_i=3\frac{\dot r_i^2}{r_i}-\frac{v_i^2}{r_i}-\frac{q_i}{m_i}\frac{r_i^4}{a^4}\frac{\partial}{\partial r_i}\left[\frac{r_i}{a}\phi'(r_i)\right]$. From Eq. (15), $\ddot r_i=3\frac{\dot r_i^2}{r_i}-\frac{v_i^2}{r_i}-(a'_r)_i\frac{r_i^2}{a^2}$, so that the definition Eq. (19) is justified.

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Therefore, if $F_r(r_j) = q_j E_r(r_j)$ is the electrostatic force on a particle j due to a distribution of particles located at positions r_i, then

$$F_r'(r_j) = q_j E_r'(r_j) = q_j \frac{r_j^2}{a^2}\left(\frac{1}{a}\phi'(r_j) + \frac{r_j}{a}\left.\frac{\partial \phi'}{\partial r}\right|_{r=r_j}\right)$$

is the force $F_r'(r_j)$ on the particle j due to the distribution of particles located at positions r_i'.

2. The inverted Klimontovich equation

We shall consider that the acceleration vector on equation (17), which is given by

$$\frac{d\mathbf{v}_i'}{dt} = \frac{\mathbf{F}(\mathbf{r}_i', \mathbf{v}_i', t)}{m_i},$$

is transformed to a function of $\mathbf{r}$ and $\mathbf{v}$,

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{F}'(\mathbf{r}, \mathbf{v}, t)}{m_\alpha},$$

by the Delta functions in the transformed Klimontovich equation. We note, on the other hand, that the radial acceleration $(a_r')_i = \left(\frac{d\mathbf{v}_i'}{dt}\right)_r$ is given as a function of the coordinates and velocities of the particles by (6):

$$(a_r')_i(\mathbf{r}_i, \mathbf{v}_i) = 3\frac{a^2}{r_i^2}\frac{\dot{r}_i^2}{r_i} - \frac{a^2}{r_i^2}\frac{v_i^2}{r_i} - \frac{a^2}{r_i^2}\ddot{r}_i. \quad (20)$$

Therefore we can write

$$(a_r')_i(\mathbf{r}_i', \dot{\mathbf{v}}_i') = \frac{F_r(r_i', \dot{r}_i', t)}{m_i}. \quad (21)$$

By means of the formation of a product along with the Delta functions of N_α', we combine (20) and (21) to obtain the radial acceleration

$$\ddot{r} = 3\frac{\dot{r}^2}{r} - \frac{v^2}{r} - \frac{r^2}{a^2}a_r = 3\frac{\dot{r}^2}{r} - \frac{v^2}{r} - \frac{r^2}{a^2}\frac{F_r'(r, \dot{r}, t)}{m_\alpha},$$

where the radial component of the force is

$$F_r' = q_\alpha \frac{r^2}{a^2}\frac{\partial}{\partial r}\left[\frac{r}{a}\phi'(r)\right].$$

Similarly, the orbital components of the generalized accelerations are

$$\ddot{\theta} = \frac{2}{r}\dot{r}\dot{\theta} + \dot{\varphi}^2 \sin\theta\cos\theta + \frac{r^2}{a^2}\frac{a_\theta}{r},$$
$$\ddot{\varphi} = \frac{2}{r}\dot{r}\dot{\varphi} - 2\dot{\theta}\dot{\varphi}\cot\theta + \frac{r^2}{a^2}\frac{a_\varphi}{r\sin\theta}.$$

The inverted Klimontovich equation is therefore given by

$$\begin{aligned}&\frac{\partial N_\alpha'}{\partial t} + \dot{r}\frac{\partial N_\alpha'}{\partial r} + \dot{\theta}\frac{\partial N_\alpha'}{\partial \theta} + \dot{\varphi}\frac{\partial N_\alpha'}{\partial \varphi}\\ &+ \left\{3\frac{\dot{r}^2}{r} - \frac{v^2}{r} - \frac{q_\alpha}{m_\alpha}\frac{r^4}{a^4}\frac{\partial}{\partial r}\left[\frac{r}{a}\phi'(r)\right]\right\}\frac{\partial N_\alpha'}{\partial \dot{r}} + 12\frac{\dot{r}}{r}N_\alpha'\\ &+ \left(\frac{2}{r}\dot{r}\dot{\theta} + \dot{\varphi}^2\sin\theta\cos\theta + \frac{r^2}{a^2}\frac{a_\theta}{r}\right)\frac{\partial N_\alpha'}{\partial \dot{\theta}}\\ &+ \left(\frac{2}{r}\dot{r}\dot{\varphi} - 2\dot{\theta}\dot{\varphi}\cot\theta + \frac{r^2}{a^2}\frac{a_\varphi}{r\sin\theta}\right)\frac{\partial N_\alpha'}{\partial \dot{\varphi}} = 0. \quad (22)\end{aligned}$$

III. THE STATISTICAL FUNCTIONS

The probability that a plasma particle is at the state $\mathbf{X}_{\alpha i}$ in the range $d\mathbf{X}_{\alpha i}$ should be equal to the one that its image is at $\mathbf{X}_{\alpha i}'$ in the range $d\mathbf{X}_{\alpha i}'$, where $d\mathbf{X}_{\alpha i}' = \frac{a^{12}}{r_{\alpha i}{}^{12}}d\mathbf{X}_{\alpha i}$. Therefore we should have

$$\begin{aligned}&\int F_N(\mathbf{X}_{\alpha 1}, \ldots, \mathbf{X}_{\alpha \bar{N}_\alpha})d\mathbf{X}_{\alpha 1}\ldots d\mathbf{X}_{\alpha \bar{N}_\alpha}\\ &= \int F_N(\mathbf{X}_{\alpha 1}', \ldots, \mathbf{X}_{\alpha \bar{N}_\alpha}')d\mathbf{X}_{\alpha 1}'\ldots d\mathbf{X}_{\alpha \bar{N}_\alpha'}' = 1.\end{aligned}$$

The total number of plasma particles is

$$\bar{N}_\alpha = \int N_\alpha(\mathbf{r}, \mathbf{v}, t)d\mathbf{X}, \quad (23)$$

where $d\mathbf{X} = d\mathbf{r}d\mathbf{v}$ is *not* taken over particle trajectories. We note that, by letting $r \to \frac{a^2}{r}$ in the integral (23), it becomes

$$\int N_\alpha(\frac{a^2}{r}\hat{r}, \mathbf{v}, t)\frac{a^{12}}{r^{12}}d\mathbf{X} = \int N_\alpha'(\mathbf{r}, \mathbf{v}, t)d\mathbf{X} = \bar{N}_\alpha'.$$

Therefore $\bar{N}_\alpha' = \bar{N}_\alpha$, or, the number of image particles is equal to the number of plasma particles, as expected.

Now we directly transform the average value of $N_\alpha(\mathbf{r}, \mathbf{v}, t)$ taken over the plasma particles trajectories,

$$\langle N_\alpha(\mathbf{r}, \mathbf{v}, t)\rangle = \int F_N N_\alpha(\mathbf{r}, \mathbf{v}, t)d\mathbf{X}_{all},$$

where F_N is the probability density function. We obtain, by using equation (14) and by considering $\bar{N}_\alpha' = \bar{N}_\alpha$ since there is a one to one correspondence between a plasma particle and its image,

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$$\left\langle N_\alpha(\frac{a^2}{r}\hat{r},\mathbf{v},t)\right\rangle = \int F_N \sum_{i=1}^{\bar{N}'_\alpha} \frac{r'_{\alpha i}{}^{12}}{a^{12}}\delta(\mathbf{r}-\mathbf{r}'_{\alpha i})\delta(\mathbf{v}-\mathbf{v}'_{\alpha i})d\mathbf{X}_{all}$$
$$= \int F_N \left[\delta(\mathbf{r}-\mathbf{r}'_{\alpha 1})\delta(\mathbf{v}-\mathbf{v}'_{\alpha 1})\frac{a^{12}}{r_{\alpha 1}{}^{12}}d\mathbf{r}_{\alpha 1}d\mathbf{v}_{\alpha 1}\ldots d\mathbf{r}_{\bar{N}_\alpha}d\mathbf{v}_{\bar{N}_\alpha}+\cdots+\delta(\mathbf{r}-\mathbf{r}'_{\bar{N}'_\alpha})\delta(\mathbf{v}-\mathbf{v}'_{\bar{N}'_\alpha})d\mathbf{r}_{\alpha 1}d\mathbf{v}_{\alpha 1}\ldots\frac{a^{12}}{r_{\bar{N}'_\alpha}{}^{12}}d\mathbf{r}_{\bar{N}_\alpha}d\mathbf{v}_{\bar{N}_\alpha}\right]$$
$$= \int F_N \left[\delta(\mathbf{r}-\mathbf{r}'_{\alpha 1})\delta(\mathbf{v}-\mathbf{v}'_{\alpha 1})d\mathbf{r}'_{\alpha 1}d\mathbf{v}'_{\alpha 1}\ldots d\mathbf{r}_{\bar{N}_\alpha}d\mathbf{v}_{\bar{N}_\alpha}+\cdots+\delta(\mathbf{r}-\mathbf{r}'_{\bar{N}'_\alpha})\delta(\mathbf{v}-\mathbf{v}'_{\bar{N}'_\alpha})d\mathbf{r}_{\alpha 1}d\mathbf{v}_{\alpha 1}\ldots d\mathbf{r}'_{\bar{N}_\alpha}d\mathbf{v}'_{\bar{N}_\alpha}\right]$$
$$= \int \left[F_N(\mathbf{r},\mathbf{v},\ldots,\mathbf{r}_{\bar{N}_\alpha},\mathbf{v}_{\bar{N}_\alpha})d\mathbf{r}_{\alpha 2}d\mathbf{v}_{\alpha 2}\ldots d\mathbf{r}_{\bar{N}_\alpha}d\mathbf{v}_{\bar{N}_\alpha}+\cdots+F_N(\mathbf{r}_{\alpha 1},\mathbf{v}_{\alpha 1},\ldots,\mathbf{r},\mathbf{v})d\mathbf{r}_{\alpha 1}d\mathbf{v}_{\alpha 1}\ldots d\mathbf{r}_{\bar{N}_\alpha - 1}d\mathbf{v}_{\bar{N}_\alpha - 1}\right]$$
$$= \bar{N}_\alpha \int F_N(\mathbf{r},\mathbf{v},\ldots,\mathbf{r}_{N_\alpha},\mathbf{v}_{N_\alpha})d\mathbf{r}_{\alpha 2}d\mathbf{v}_{\alpha 2}\ldots d\mathbf{r}_{\bar{N}_\alpha}d\mathbf{v}_{\bar{N}_\alpha} = \bar{n}_\alpha f_\alpha(\mathbf{r},\mathbf{v},t) = \langle N_\alpha(\mathbf{r},\mathbf{v},t)\rangle\,.$$

Therefore, the average value of N_α is invariant under inversion in a sphere, yielding the same one-particle distribution function. This suggests a statistical equivalence of the plasma with its image, where

$$f_\alpha(\mathbf{r}_1,\mathbf{v}_1,t) = V\int F_N(\mathbf{X}_{\alpha 1},\ldots,\mathbf{X}_{\alpha\bar{N}_\alpha})d\mathbf{X}_{\alpha 2}\ldots d\mathbf{X}_{\alpha\bar{N}_\alpha} \quad (24)$$

is the one particle distribution function.

Also we should verify the average value of N'_α taken over $d\mathbf{X}'_{all}$:

$$\langle N'_\alpha(\mathbf{r},\mathbf{v},t)\rangle = \int F_N \sum_{i=1}^{\bar{N}'_\alpha}\delta(\mathbf{r}-\mathbf{r}'_i)\delta(\mathbf{v}-\mathbf{v}'_i)d\mathbf{X}'_{all}$$
$$= \bar{N}'_\alpha \int F_N(\mathbf{r},\mathbf{v},\ldots,\mathbf{r}'_{\bar{N}'_\alpha},\mathbf{v}'_{\bar{N}'_\alpha})d\mathbf{r}'_2 d\mathbf{v}'_2\ldots d\mathbf{v}'_{\bar{N}'_\alpha}$$
$$= \bar{n}'_\alpha f'_\alpha(\mathbf{r},\mathbf{v},t) = \langle N'_\alpha(\mathbf{r},\mathbf{v},t)\rangle\,,$$

where

$$V'\int F_N(\mathbf{X}'_{\alpha 1},\ldots,\mathbf{X}_{\alpha\bar{N}'_\alpha})d\mathbf{X}'_{\alpha 2}\ldots d\mathbf{X}'_{\alpha\bar{N}'_\alpha} = f'_\alpha(\mathbf{r}'_1,\mathbf{v}'_1,t) \quad (25)$$

is the primed one particle distribution function.

By inverting coordinates and integrating both (24) and (25) over $d\mathbf{X}'$ and $d\mathbf{X}$ respectively, we obtain

$$\int f_\alpha(\mathbf{r}',\mathbf{v}',t)\left(\frac{a}{r}\right)^{12} d\mathbf{X} = \frac{V}{V'}\int f'_\alpha(\mathbf{r},\mathbf{v},t)d\mathbf{X}.$$

Therefore, for an arbitrary range of integration in $d\mathbf{X}$ and since $\bar{N}_\alpha = \bar{N}'_\alpha$,

$$f_\alpha(\mathbf{r}',\mathbf{v}',t) = \frac{\bar{n}'_\alpha}{\bar{n}_\alpha}\frac{r^{12}}{a^{12}}f'_\alpha(\mathbf{r},\mathbf{v},t). \quad (26)$$

Expression (26) is the statistical equivalent of the microscopic expression (16).

A. The statistical kinetic equation for f'_α

The average of equation (22) over $d\mathbf{X}'_{all}$, considering the radial motion only to simplify the analysis, yields

$$\frac{\partial f'_\alpha}{\partial t} + \dot{r}\frac{\partial f'_\alpha}{\partial r} + \frac{1}{\bar{n}'_\alpha}\left\langle\left\{2\frac{\dot{r}^2}{r} - \frac{q_\alpha}{m_\alpha}\frac{r^4}{a^4}\frac{\partial}{\partial r}\left[\frac{r}{a}\phi'(r)\right]\right\}\frac{\partial N'_\alpha}{\partial \dot{r}}\right\rangle + 12\frac{\dot{r}}{r}f'_\alpha = 0, \quad (27)$$

which is the exact kinetic equation. The integration of this equation in the range $r < a$, along with the Poisson's equation for the inverted average potential, is equivalent to the non primed system of equations in the range $r > a$.

Let us now make some development upon the Poisson's equation and average inverted potential.

B. The Poisson's equation for the average inverted potential

In order to obtain the Poisson's equation in terms of the average potential, let us verify how the microscopic potential ϕ, given as an integral over the continuous space, in the form

$$\phi(\mathbf{r},t) = \sum_\alpha q_\alpha \int \frac{N_\alpha(\mathbf{r}_1,\mathbf{v}_1,t)}{|\mathbf{r}-\mathbf{r}_1|}d\mathbf{X}_1, \quad (28)$$

transforms under a^2/r. Here we have used a numbered notation to identify the variable of integration, so that it can take the form $\mathbf{X}_{1,2,\ldots}$ or $\mathbf{X}'_{1,2,\ldots}$ over the inverted space.

The factor

$$|\mathbf{r}-\mathbf{r}_1| = \left(r^2 + r_1^2 - 2rr_1\cos\gamma\right)^{1/2}$$

transforms like

$$\left|\frac{a^2}{r}\hat{r}-\mathbf{r}_1\right| = \left(\frac{a^4}{r^2}+r_1^2-2\frac{a^2}{r}r_1\cos\gamma\right)^{1/2}$$
$$= \frac{r_1}{r}\left(\frac{a^4}{r_1^2}+r^2-2\frac{a^2}{r_1}r\cos\gamma\right)^{1/2}.$$

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Here we see that the transformation $r \to \frac{a^2}{r}$ also implies a transformation $r_1 \to \frac{a^2}{r_1}$ on the variable of integration. Therefore, we obtain from (28)

$$\begin{aligned}
&\phi(\frac{a^2}{r}\hat{r},t) \\
&= \sum_\alpha q_\alpha \int \frac{r_1^{12}}{a^{12}} \frac{N'_\alpha(\mathbf{r}_1,\mathbf{v}_1,t)}{\frac{r_1}{r}\left(\frac{a^4}{r_1^2}+r^2-2\frac{a^2}{r_1}r\cos\gamma\right)^{1/2}} \frac{a^{12}}{r_1^{12}} d\mathbf{X}_1 \\
&= \frac{r}{a}\sum_\alpha q_\alpha \int \frac{a}{r_1} \frac{N'_\alpha(\mathbf{r}_1,\mathbf{v}_1,t)}{\left(\frac{a^4}{r_1^2}+r^2-2\frac{a^2}{r_1}r\cos\gamma\right)^{1/2}} d\mathbf{X}_1 = \frac{r}{a}\phi'(\mathbf{r},t),
\end{aligned} \tag{29}$$

where

$$\phi'(\mathbf{r},t) = \sum_\alpha q_\alpha \int \frac{a}{r_1} \frac{N'_\alpha(\mathbf{r}_1,\mathbf{v}_1,t)}{\left(\frac{a^4}{r_1^2}+r^2-2\frac{a^2}{r_1}r\cos\gamma\right)^{1/2}} d\mathbf{X}_1$$

is the integral form of ϕ'.

The Poisson's equation for the microscopic potential is

$$\nabla^2\phi = -4\pi\sum_\alpha q_\alpha \int N_\alpha(\mathbf{r},\mathbf{v},t)d\mathbf{v}$$

which transforms to

$$\nabla^2\phi' = -4\pi\sum_\alpha q_\alpha \int \frac{r}{a}N'_\alpha(\mathbf{r},\mathbf{v},t)d\mathbf{v}, \tag{30}$$

where we have used (4) and (16).

Averaging (30) over $d\mathbf{X}'_{all}$ we obtain

$$\begin{aligned}
&\nabla^2\Phi' \\
&= -4\pi\sum_\alpha q_\alpha \int \left[\int F_N \sum_{i=1}^{\bar{N}'_\alpha} \frac{r'_i}{a}\delta(\mathbf{r}-\mathbf{r}'_i(t))\delta(\mathbf{v}-\mathbf{v}'_i(t))d\mathbf{X}'_{all}\right] d\mathbf{v} \\
&= -4\pi\frac{r}{a}\sum_\alpha q_\alpha \bar{n}'_\alpha \int f'_\alpha(\mathbf{r},\mathbf{v},t)d\mathbf{v} \qquad (31)\\
&= -4\pi\frac{r}{a}\sum_\alpha q_\alpha \bar{n}_\alpha \int \left(\frac{a}{r}\right)^6 f_\alpha(\mathbf{r}',\mathbf{v}',t)d\mathbf{v}' = \frac{a^5}{r^5}\nabla^2\Phi,
\end{aligned}$$

which is in accordance with (3) and where we have used (26), and the average potential $\Phi'(\mathbf{r},t)$ is given by

$$\begin{aligned}
&\langle \phi'(\mathbf{r},t)\rangle = \Phi'(\mathbf{r},t) \\
&= \int F_N \left[\sum_\alpha q_\alpha \int \frac{a}{r_1} \frac{N'_\alpha(\mathbf{r}_1,\mathbf{v}_1,t)}{\left(\frac{a^4}{r_1^2}+r^2-2\frac{a^2}{r_1}r\cos\gamma\right)^{1/2}} d\mathbf{X}_1\right] d\mathbf{X}'_{all} \\
&= \sum_\alpha q_\alpha \bar{n}'_\alpha \int \frac{a}{r_1} \frac{f'_\alpha(\mathbf{r}_1,\mathbf{v}_1,t)}{\left(\frac{a^4}{r_1^2}+r^2-2\frac{a^2}{r_1}r\cos\gamma\right)^{1/2}} d\mathbf{X}_1.
\end{aligned}$$

C. Fluctuations

Introducing the fluctuations

$$\delta N'_\alpha(\mathbf{r},\mathbf{v},t) = N'_\alpha(\mathbf{r},\mathbf{v},t) - \bar{n}'_\alpha f'_\alpha(\mathbf{r},\mathbf{v},t), \tag{32}$$

$$\delta\phi'(\mathbf{r},t) = \phi'(\mathbf{r},t) - \Phi'(\mathbf{r},t), \tag{33}$$

and substituting to (27) we obtain

$$\frac{\partial f'_\alpha}{\partial t} + \dot{r}\frac{\partial f'_\alpha}{\partial r} + \left\{2\frac{\dot{r}^2}{r} - \frac{q_\alpha}{m_\alpha}\frac{r^4}{a^4}\frac{\partial}{\partial r}\left[\frac{r}{a}\Phi'\right]\right\}\frac{\partial f'_\alpha}{\partial \dot{r}} + 12\frac{\dot{r}}{r}f'_\alpha = \left\langle \frac{q_\alpha}{m_\alpha \bar{n}'_\alpha}\frac{r^4}{a^4}\frac{\partial}{\partial r}\left[\frac{r}{a}\delta\phi'\right]\frac{\partial \delta N'_\alpha}{\partial \dot{r}}\right\rangle, \tag{34}$$

where the conditions

$$\left\langle \frac{\partial}{\partial r}\left[\frac{r}{a}\delta\phi'\right]\right\rangle = 0,$$

$$\left\langle \frac{\partial \delta N'_\alpha}{\partial \dot{r}}\right\rangle = 0,$$

follow from (32) and (33).

D. The transformed Vlasov equation and the final system of equations

The transformed Vlasov equation is finally obtained from (34),

$$\frac{\partial f'_\alpha}{\partial t} + \dot{r}\frac{\partial f'_\alpha}{\partial r} + \left\{2\frac{\dot{r}^2}{r} - \frac{q_\alpha}{m_\alpha}\frac{r^4}{a^4}\frac{\partial}{\partial r}\left[\frac{r}{a}\Phi'\right]\right\}\frac{\partial f'_\alpha}{\partial \dot{r}} + 12\frac{\dot{r}}{r}f'_\alpha = 0. \tag{35}$$

From equation (19) it is clear that we can make the replacement

$$\frac{r^2}{a^2}\frac{\partial}{\partial r}\left[\frac{r}{a}\Phi'\right] = \mathfrak{E}'_r(r), \tag{36}$$

where $\mathfrak{E}'_r(r)$ is the inverted average field, in the Vlasov equa-

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tion (35) and in the Poisson's equation (31), considering

$$\nabla^2\Phi' = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi') = \frac{a}{r}\frac{\partial}{\partial r}\left[\frac{a^2}{r^2}\mathfrak{E}'_r(r)\right].$$

Therefore we obtain the system of equations

$$\frac{\partial f'_\alpha}{\partial t} + \dot{r}\frac{\partial f'_\alpha}{\partial r} + \left\{2\frac{\dot{r}^2}{r} - \frac{q_\alpha}{m_\alpha}\frac{r^2}{a^2}\mathfrak{E}'_r\right\}\frac{\partial f'_\alpha}{\partial \dot{r}} + 12\frac{\dot{r}}{r}f'_\alpha = 0, \quad (37)$$

$$\frac{\partial}{\partial r}\left[\frac{a^2}{r^2}\mathfrak{E}'_r\right] = -4\pi\frac{r^2}{a^2}\sum_\alpha q_\alpha \bar{n}'_\alpha \int f'_\alpha(\mathbf{r},\mathbf{v},t)d\mathbf{v}. \quad (38)$$

These two equations form the basis for the solution of an electrostatic radial problem. In the present work, two examples on the obtention of the plasma potential for given plasma distributions will be provided in Sec. IV.

Given the already developed inverted equations, we are in condition to enounce that, *If f_α is a solution of the Vlasov equation, then f'_α given by Eq.* (26) *is a solution of Eq.* (37).

IV. APPLICATION OF THE FORMALISM

We can apply the inverted transformed quantities by following two approaches.

The first approach is to replace the original system by the transformed one and obtain a solution for this one. If an analytical solution is achieved, the reverse transformation can be applied and the original quantities evaluated. In Section IV B we formulate an example of this approach. Section IV B also has the additional purpose of certifying the correctness of the developed inverted equations so far, since a known potential is obtained.

The second approach is to use the transformed equations to extend the plasma inside the sphere, so that the integration can be made in the whole space. This should be particularly useful for the study of plasma waves under Fourier integration.

A. Stationary states

As an example on how a solution of stationary states transforms under inversion, let us take the Boltzmann distribution for radial motion

$$f_\alpha(\mathbf{r},\mathbf{v}) = A_\alpha \exp\left(-\frac{\dot{r}^2}{2v_\alpha^2} - \frac{q_\alpha}{m_\alpha v_\alpha^2}\Phi(r)\right)\frac{1}{r^2\sin\theta}\delta(\dot{\theta})\delta(\dot{\varphi}),$$

where A_α is the normalization constant and v_α is the thermal velocity, $v_\alpha = (k_B T_\alpha/m_\alpha)^{1/2}$, with k_B being the Boltzmann constant and T_α the temperature associated with the radial kinetic energy of the particles of species α. The function Φ is the average potential. Under inversion f_α transforms accordingly to

$$f_\alpha(\mathbf{r}',\mathbf{v}') = A_\alpha \exp\left(-\frac{a^4}{r^4}\frac{\dot{r}^2}{2v_\alpha^2} - \frac{q_\alpha}{m_\alpha v_\alpha^2}\frac{r}{a}\Phi'(r)\right) \times \frac{r^2}{a^4\sin\theta}\delta(\dot{\theta})\delta(\dot{\varphi})$$

so that, from Eq. (26),

$$f'_\alpha(\mathbf{r},\mathbf{v}) = A_\alpha\frac{\bar{n}_\alpha}{\bar{n}'_\alpha}\frac{a^{12}}{r^{12}}\exp\left(-\frac{a^4}{r^4}\frac{\dot{r}^2}{2v_\alpha^2} - \frac{q_\alpha}{m_\alpha v_\alpha^2}\frac{r}{a}\Phi'(r)\right) \times \frac{r^2}{a^4\sin\theta}\delta(\dot{\theta})\delta(\dot{\varphi}). \quad (39)$$

Let $f_\alpha(r,\dot{r}) = \int f_\alpha(\mathbf{r},\mathbf{v})r^2\sin\theta d\dot{\theta}d\dot{\varphi}$. Therefore, $f'_\alpha(r,\dot{r})$ should be a solution of Eq. (35). Indeed, by taking $(\partial f'_\alpha/\partial t) = 0$, we obtain

$$\dot{r}\frac{\partial f'_\alpha}{\partial r} + \left\{2\frac{\dot{r}^2}{r} - \frac{q_\alpha}{m_\alpha}\frac{r^4}{a^4}\frac{\partial}{\partial r}\left[\frac{r}{a}\Phi'\right]\right\}\frac{\partial f'_\alpha}{\partial \dot{r}} + 12\frac{\dot{r}}{r}f'_\alpha = 0. \quad (40)$$

By calculating the explicit partial derivatives out of (39),

$$\frac{\partial f'_\alpha}{\partial r} = -\frac{12}{r}f'_\alpha + \left(2\frac{a^4}{r^5}\frac{\dot{r}^2}{v_\alpha^2} - \frac{q_\alpha}{m_\alpha v_\alpha^2}\frac{\partial}{\partial r}\left[\frac{r}{a}\Phi'(r)\right]\right)f'_\alpha,$$

$$\frac{\partial f'_\alpha}{\partial \dot{r}} = -\frac{a^4}{r^4}\frac{\dot{r}}{v_\alpha^2}f'_\alpha,$$

and substituting them into (40), the steady state solution (39) is verified.

This specific example does not reduce the generality of Eq. (37), which should be valid for any Vlasov equilibrium state.

B. Debye-Hückel potential

Let ϕ be the microscopic potential in the neighborhood of a conducting sphere surrounded by a plasma. The value of ϕ at any r in space is given by the sum of the discrete potentials of each plasma particle plus its image. It is clear, therefore, that the average potential inside the sphere is given by Eq. (31), since the average of all delta distributions for the plasma particles is zero inside the sphere.

We start by performing the velocity integration in (31) using Eq. (39) with normalization constant $A_\alpha = (1/2\pi)^{1/2}v_\alpha$, considering an electron-ion plasma, and obtain the linearised equation

$$\begin{aligned}&\frac{\partial^2\Psi'}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial\Psi'}{\partial\rho}\\ &= \frac{\ell^2}{\rho^5}\left\{\left[\left(1-\mathrm{erf}\left(\sqrt{-\rho\Psi'}\right)\theta(-\Psi')\right)e^{\rho\Psi'}\right.\right.\\ &\left.\left. - \eta_i\left(1-\mathrm{erf}\left(\sqrt{\tau_i\rho\Psi'}\right)\theta(\tau_i\Psi')\right)e^{-\tau_i\rho\Psi'}\right]\right\}\\ &\approx \frac{\ell^2}{\rho^4}\Psi',\end{aligned} \quad (41)$$

which is written in terms of the dimensionless quantities

$$\Psi = \frac{e\Phi}{k_B T_e},\quad \boldsymbol{\rho} = \frac{\mathbf{r}}{a},\quad \ell = \frac{a}{\lambda_e},\quad \eta_\alpha = \frac{\bar{n}_\alpha}{\bar{n}_e},\quad \tau_\alpha = \frac{T_e}{T_\alpha},$$

where e is the fundamental charge, the Debye length λ_e is defined by $(k_B T_e/4\pi e^2\bar{n}_e)^{1/2}$ and $\bar{n}_\alpha$ is the density at infinity. In obtaining (41) we have made $\eta_i = 1$ and $\tau_i \gg 1$.

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The general solution of Eq. (41) is given by $\Psi'(\rho) = k_2 \cosh(\ell/\rho) - ik_1 \sinh(\ell/\rho)$, where k_1 and k_2 are constants. If the boundary conditions at $\rho = 1$ are $\Psi'(1)$ and $(\partial\Psi'/\partial\rho)_{\rho=1} = \ell\Psi'(1)$, then

$$\Psi'(\rho) = \Psi'(1) \exp\left[-\left(\frac{1}{\rho} - 1\right)\ell\right]. \qquad (42)$$

This is the inverted Debye-Hückel potential under transformation (3), and, therefore, this certifies the correctness of the power law given in the right hand side of the linearised equation (41), and also the correctness of Eqs. (31), (35) and (39). We can also use (42) and its inverse to verify that[2]

$$\int_{0\le r\le a} \left(\nabla^2\Phi'\right) d\mathbf{r} = \int_{r\ge a} \left(\nabla^2\Phi\right) \frac{a}{r} d\mathbf{r}. \qquad (43)$$

C. The induced charge in a conducting sphere

A simple application of the formalism is to obtain the total induced charge by the plasma on a conducting sphere. Using the charge density from Poisson's equation (31), the total induced charge is given by

$$Q' = \int \frac{r}{a} \sum_\alpha q_\alpha \bar{n}'_\alpha \int f'_\alpha(\mathbf{r},\mathbf{v}) d\mathbf{v} d\mathbf{r} \quad , \quad 0 \le r \le a. \qquad (44)$$

If the plasma potential is the Debye-Hückel potential, then the induced charge number is

$$Z' = -\frac{\ell^2}{\delta} \int_0^1 \frac{\Psi'}{\rho^2} d\rho = -\frac{\ell\Psi'(1)}{\delta} = -\frac{\ell}{\delta_e} \left(\frac{\partial\Psi'}{\partial\rho}\right)_{\rho=1}, \qquad (45)$$

where $\delta = e^2/ak_BT_e = e^2/\ell\lambda_e k_BT_e = \delta_e/\ell$. We see that the expression (44) yields the charge number $Z' = 4\pi a^2 \sigma'/e$, where $\sigma' = -(1/4\pi)(\partial\Phi'/\partial r)_{r=a}$ is the portion of the total surface charge density due to charge induction by the plasma. The total charge density defined by $4\pi\sigma_T = \mathfrak{E}'_r(a)$ yields the total charge number Z_T on the surface,

$$Z_T = \frac{\Psi'(1)}{\delta} - Z' \qquad (46)$$

where we have used the expression of the electric field (36).

Expression (46), which was obtained from the transformed quantities, is equivalent to the expression $C\Phi(a) = Q$ commonly used in dusty plasmas charging theories for spherical grains[7,8], where Q is the dust grain charge and C is the capacitance given by $a^2(1/a + 1/\lambda_e)$. When $\ell \ll 1$, the approximation $C = a$ is equivalent to $Z' = 0$. Expression (46) makes it clear also that the second term, a^2/λ_e, of the capacitance is due to the plasma induced charge on the conducting sphere.

One may argue that the use of Eqs. (36), (44), (45) and (46) is not necessary, since the same expression for the total charge is provided by the potential Ψ, $Z_T = (1+\ell)\Psi(1)/\delta$, by requiring continuity of the potential at $\rho = 1$. Not to mention the simplicity of just using the expression $C\Phi(a) = Q$. However, there may be plasma distributions for which the analytical expression of the potential is not known and so is not the capacitance. For this case, if a numerical solution of the potential is provided, then the induced charge can be estimated by using Eq. (45).

V. FINAL REMARKS

In the present analysis we have developed the *inversion in a sphere* transformations of several microscopic functions and microscopic equations of a plasma, which were necessary for the outcome of the inverted Klimontovich equation (22). We have also obtained the associated statistical functions necessary for the outcome of the inverted radial Vlasov equation (35), which was the main purpose of the present work. We have found that the transformed Vlasov equation (35) is not invariant under the transformation (1), unlike the Poisson's equation which is invariant under the same transformation[1]. However, in general, if f_α is a solution of the Vlasov equation, then f'_α given by (26) is a solution of Eq. (35).

We have shown, as an example, that the Boltzmann distribution, which is a stationary state of the Vlasov equation, can be inverted to become a solution of (35). We have certified the correctness of our development in Sec. IV by the application of the Vlasov-Poisson's inverted equations to obtain the inverted Debye-Hückel potential. Also we have shown that the inverted field at the surface permits to distinguish the induced plasma charge on the sphere from the total charge on it, and that the expression for the total charge on the sphere is equal to the one used in dusty plasmas charging theories.

The usefulness of the transformed form (35) should be proven in practice: if there are plasma physics configurations for which it is easier to integrate the inverted Vlasov equation for $r < a$ than it is to integrate the Vlasov equation for $r > a$. To foreseen the number of applications on the field of plasma physics is beyond the horizon our sight can reach at the current date, but we believe that the present theoretical analysis is quite general to electrostatic plasma physics problems.

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